J. Fluid Mech. (1982), *vol.* 123, *pp.* 283–301
Printed in Great Britain

Patterns of convection in spherical shells. Part 2

By F. H. BUSSE†
Max-Planck-Institut für Astrophysik, Garching near Munich

AND N. RIAHI‡
Institute of Geophysics and Planetary Physics, UCLA

(Received 10 February 1981 and in revised form 17 November 1981)

The analysis by Busse (1975) of preferred patterns of convection in spherical shells is extended to include the case of odd degrees l of spherical harmonics. In the general part of the paper only the property of spherical symmetry of the basic state is used. The results are thus applicable to all bifurcation problems with spherical symmetry. Except in the case $l = 1$ a pattern degeneracy of the linear problem exists, which is partly removed by the solvability conditions that are generated when nonlinear terms are taken into account as perturbations. In each of the cases l considered so far, at least l physically different solutions have been found. The preferred solution among l existing ones is determined for $l \geqslant 2$ by a stability analysis. In the case $l = 3$ emphasized in this paper the axisymmetric solution is found to be always unstable, and the solution of tetrahedronal symmetry appears to be generally preferred. The latter result is rigorously established in the special case of a thin layer with nearly insulating boundaries treated in the second part of the paper.

1. Introduction

The problem of convection in spherical shells with spherically symmetric physical conditions is of interest from several points of view. The early work on linear aspects of the problem, which is conveniently reviewed in Chandrasekhar's (1961) monograph, has been motivated by the hypothesis of convection flow in the Earth's mantle. The increasing evidence for solid-state convection in the earth and in other terrestial planets, at least throughout parts of their thermal history, has stimulated a number of nonlinear analyses of the problem of convection in spherical shells (Young 1974; Busse 1975; Zebib, Schubert & Straus 1980; Schubert & Zebib 1980). But there are other applications as well. The physical conditions of convection zones in stars are spherically symmetric if the rotation rate is sufficiently low. Moreover the results obtained in the theory of convection can be applied to other cases of instability of a spherically symmetric system such as the buckling of spherical shells. In fact any problem involving a symmetry-breaking bifurcation from a spherically symmetric state leads to the same problems of pattern selection as those that can be discussed in the case of convection.

From the mathematical point of view, convection in spherically symmetric layers is of special interest because of the finite degeneracy of the bifurcation problem. Since the degree of pattern degeneracy is closely related to the degree l of spherical

† Permanent address: Institute of Geophysics and Planetary Physics, University of California at Los Angeles.
‡ Present address: University of Illinois at Urbana–Champaign.

harmonics that is preferred according to the linear theory of the problem, the degree of pattern degeneracy varies primarily with the radius ratio of the boundaries of the shell, and a rich variety of bifurcation problems can be studied in terms of the dependence on the latter parameter. The case $l = 2$ has received most of the attention so far (Chossat 1979; Golubitsky & Schaeffer 1982), but other cases may prove to be of no lesser mathematical interest once the appropriate group-theoretical methods are available (Sattinger 1980).

The main mathematical problem of convection in spherical shells is the determination of independent solutions of the basic equations. Because most of the solutions given in the form of the usual superposition of spherical harmonics are transformations of each other, it is a non-trivial task to enumerate those solutions that cannot be transformed into each other by rotation on the sphere. In this paper sets of independent solutions will be derived, but a rigorous proof for the completeness of the sets is not available in most cases. Of primary physical interest are the stable solutions of the problem. For this reason a stability analysis is carried out in this paper in addition to the determination of the steady solutions of the problem.

In order to keep the discussion of the problem as general as possible, no special assumptions about physical properties will be made (except for the consideration of a special case at the end of the paper). The main restriction of the analysis is that the Rayleigh number must be close to its critical value such that the amplitude of convection is small and an expansion in powers of the amplitude is feasible. In this respect the paper is an extension of an earlier work of Busse (1975, hereinafter referred to as B75). By carrying the perturbation analysis of that paper to the third order in the amplitude of convection it becomes possible to derive some general properties of cases with odd degrees l of the spherical harmonics. In B75 preferred solutions have been derived only in the case of even l. In particular, it was shown that convection modes exhibiting the symmetries of four of the five Platonic bodies are preferred in the cases of $l = 4$ and $l = 6$. In this paper it will be demonstrated that convection with tetrahedronal symmetry is preferred in the case $l = 3$.

The paper starts with a description of the basic equations and a formulation of the perturbation analysis in §2. To emphasize the general properties of the analysis, dependent variables are combined into a vector variable, and matrix operators are introduced. Steady solutions are discussed from the general point of view in §3, and evaluated for particular values of the degree l of spherical harmonics in §5. Similarly, a general stability analysis of the steady solutions is described in §4, while an evaluation of the expressions for the growth rates of disturbances for particular values of l is postponed until §6. In §7 the limit of a thin shell with nearly insulating boundaries is assumed, which permits the derivation of explicit analytical expressions. The paper closes with some concluding remarks in §8.

2. Mathematical formulation of the problem

We consider a spherical fluid layer of thickness h which is bounded by two concentric spherical surfaces with radii $r_0 h$ and $(r_0+1)h$. The fluid is subject to a spherically symmetric gravity force and contains a spherically symmetric distribution of heat sources. The properties of the core inside the fluid layer are also assumed to exhibit spherical symmetry. Because of its symmetry the problem permits a steady static solution of pure conduction with a temperature drop ΔT across the fluid layer.

Using h, h^2/κ, and ΔT as scales for length, time and temperature respectively, where κ is the thermal diffusivity, the basic equations in the Boussinesq approximation can be written in the dimensionless form given in B75. Here we proceed by introducing the general representation for the solenoidal velocity vector $\mathbf{u}$

$$\mathbf{u} = \nabla \times (\nabla \times \mathbf{r}\Phi) + \nabla \times \mathbf{r}\psi. \tag{2.1}$$

By scalar-multiplying the curl of the curl of the equation of motion by $\mathbf{r}$ the following scalar equation for Φ is obtained:

$$\left(\nabla^2 - P^{-1}\frac{\partial}{\partial t}\right)\nabla^2 L_2 \Phi - R\hat{\gamma}(r)\, L_2 \Theta/(r_0+\tfrac{1}{2}) = P^{-1}\mathbf{r}\,.\,\nabla \times \{\nabla \times (\mathbf{u} \times (\nabla \times \mathbf{u}))\}. \tag{2.2a}$$

The heat equation for the deviation Θ of the temperature from the dimensionless static temperature distribution $T(r)$ is given by

$$\left(\nabla^2 - \frac{\partial}{\partial t}\right)\Theta + \frac{T'(r) L_2 \Phi}{r} = \mathbf{u}\,.\,\nabla\Theta. \tag{2.2b}$$

The operation L_2 represents the negative two-dimensional Laplacian on the unit sphere, i.e. in spherical co-ordinates (r, θ, ϕ)

$$L_2 \equiv -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

The Rayleigh number R and the Prandtl number P are defined by

$$R = \frac{\alpha g_0 \Delta T h^3}{\kappa\nu}, \quad P = \frac{\nu}{\kappa},$$

where α is the coefficient of thermal expansion, ν is the kinematic viscosity and g_0 is the gravitational acceleration at $r = r_0 + \frac{1}{2}$, such that the function $\hat{\gamma}(r)$ becomes unity at $r = r_0 + \frac{1}{2}$.

There is no need to consider an equation for ψ as far as the analysis is carried out in this paper.

Chandrasekhar (1961) has shown that the equation for ψ admits only decaying solutions when the nonlinear terms are neglected. Thus ψ can be at most of order ϵ^2 if ϵ is a measure of the amplitude of convection. In appendix A it is shown that ψ can be at most of order ϵ^3, since the right-hand side of the equation for ψ vanishes in order ϵ^2. This property is analogous to the corresponding property in the case of a plane layer, (Schlüter, Lortz & Busse 1965) and holds for the stability analysis as well.

To simplify the notation, Φ and Θ are combined into a two-dimensional vector $\mathbf{X}^{\mathrm{T}} \equiv \{(r_0+\frac{1}{2})\Phi,\ \Theta\}$, and the matrix operators

$$\mathbf{W} \equiv \begin{pmatrix} -\nabla^4 L_2 & 0 \\ T'(r)[r(r_0+\frac{1}{2})]^{-1} L_2 & \nabla^2 \end{pmatrix}, \quad \mathbf{U} \equiv \begin{pmatrix} 0 & \hat{\gamma}(r)\, L_2 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{V} \equiv \begin{pmatrix} -P^{-1}\nabla^2 L_2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{Q}(\mathbf{X}, \breve{\mathbf{X}}) \equiv \begin{pmatrix} P^{-1}(r_0+\frac{1}{2})\,\mathbf{r}\,.\,\nabla \times \nabla \times \{(\nabla \times (\nabla \times \mathbf{r}\Phi)) \times (\mathbf{r} \times \nabla\nabla^2\breve{\Phi})\} \\ \nabla \times (\nabla \times \mathbf{r}\Phi)\,.\,\nabla\breve{\Theta} \end{pmatrix} \tag{2.3}$$

are introduced. Equations (2.2a, b) for steady solutions Φ, Θ, can thus be written in the form

$$(\mathbf{W} + R\mathbf{U})\,.\,\mathbf{X} = \mathbf{Q}(\mathbf{X}, \mathbf{X}). \tag{2.4}$$

There is no need to specify boundary conditions at this point. It is sufficient to assume that the boundary conditions are linear and homogeneous in the variables Φ, Θ:

$$\mathbf{B}_i\,.\,\mathbf{X} = 0 \quad \text{at} \quad r = r_0, \quad \mathbf{B}_0\,.\,\mathbf{X} = 0 \quad \text{at} \quad r = r_0 + 1. \tag{2.5}$$

In order to investigate the stability of the steady solution $\mathbf{X}$ of the boundary-value problem (2.4), (2.5), infinitesimal disturbances $\tilde{\mathbf{X}}$ will be superimposed. Without losing generality it can be assumed that the time dependence of the disturbances is of the form $\exp\{\sigma t\}$, with a complex growth rate σ. The equations for the disturbances can thus be written in the form

$$(\mathbf{W}+R\mathbf{U}).\tilde{\mathbf{X}} = \sigma\mathbf{V}.\tilde{\mathbf{X}}+\mathbf{Q}(\mathbf{X},\tilde{\mathbf{X}})+\mathbf{Q}(\tilde{\mathbf{X}},\mathbf{X}). \tag{2.6}$$

Solutions of the nonlinear problem (2.4), (2.5) can be obtained by expanding the dependent variable $\mathbf{X}$ and the Rayleigh number R in powers of the amplitude ϵ of convection. Here and in the subsequent stability analysis we follow basically the approach of Schlüter *et al.* (1965) and Busse (1967) in the case of a plane layer. After introducing the power series

$$\mathbf{X} = \epsilon\mathbf{X}^{(1)}+\epsilon^2\mathbf{X}^{(2)}+\epsilon^3\mathbf{X}^{(3)}+\dots, \quad R = R^{(0)}+\epsilon R^{(1)}+\epsilon^2R^{(2)}+\dots \tag{2.7}$$

into (2.4), a hierarchy of linear equations is obtained corresponding to different powers of ϵ. Starting with terms of order ϵ these equations are solved term by term. In this paper only terms up to order ϵ^3 will be considered.

In lowest order, (2.4) yields

$$(\mathbf{W}+R^{(0)}\mathbf{U}).\mathbf{X}^{(1)} = 0, \tag{2.8}$$

which has been studied in detail by Chandrasekhar (1961). The general solution can be written in the form

$$\mathbf{X}^{(1)} = \begin{pmatrix}\Phi^{(1)}(r_0+\frac{1}{2})\\ \Theta^{(1)}\end{pmatrix} = \begin{pmatrix}f(r)\\ g(r)\end{pmatrix} w_l(\theta,\phi), \tag{2.9}$$

where

$$w_l(\theta,\phi) \equiv \sum_{m=0}^{l}(\alpha_m\cos m\phi+\beta_m\sin m\phi)\,\hat{P}_l^m(\cos\theta) \equiv \sum_{m=0}^{l}(\alpha_m Y_l^m+\beta_m\hat{Y}_l^m) \tag{2.10}$$

represents the general solution of

$$L_2 w_l = l(l+1)\,w_l. \tag{2.11}$$

The functions $\hat{P}_l^m(\cos\theta)$ differ by a factor from the commonly used associated Legendre polynomials

$$\hat{P}_l^m(x) \equiv |(2l+1)(2-\delta_{m0})(l-m)!/(l+m)!|^{\frac{1}{2}}P_l^m(x).$$

This definition ensures that

$$\langle|\hat{P}_l^m(\cos\theta)\cos m\phi|^2\rangle = \langle|\hat{P}_l^m(\cos\theta)\sin m\phi|^2\rangle = 1 \quad \text{for all} \quad m,l,$$

where the angle brackets indicate the average over the fluid shell

$$\langle\dots\rangle = \frac{1}{4\pi[r_0(r_0+1)+\frac{1}{3}]}\int_{r_0}^{r_0+1}\int_0^{2\pi}\int_0^{\pi}\dots\sin\theta\,d\theta\,d\phi\,r^2dr. \tag{2.12}$$

Using (2.11), (2.8) can be reduced to two coupled ordinary differential equations for $f(r)$ and $g(r)$. In the following it will be assumed that the functions $f(r)$ and $g(r)$ satisfying the boundary conditions (2.5) have been determined. That value of l will be assumed which minimizes the eigenvalue $R^{(0)}$. Most of the arguments used in the following will not depend on this particular choice of l. Since the eigenvalues $R^{(0)}$ for the different l are distinct in general, only a single value of l must be considered to order ϵ of the problem. The singular case when the difference between two eigenvalues $R^{(0)}$ for different values of l tends to zero requires special discussion and will not be addressed in this paper.

The arbitrariness of the coefficients α_m, β_m in expression (2.10) exhibits the $(2l+1)$-fold degeneracy of the solution of the linear problem. Not all solutions of the form (2.10) correspond to solutions of the nonlinear problem in the limit $\epsilon \rightarrow 0$. The main task of the analysis in §3 is the determination of the constraints on the coefficients. This task is complicated by the fact that together with any given solution of the form (2.10) all solutions obtained by rotations of the given solution belong to the manifold of solutions of the problem. To simplify the analysis we shall follow B75 in introducing the assumption that all solutions of the nonlinear problem (2.4), (2.5) exhibit symmetry with respect to a plane through the centre of the sphere. By identifying this plane with the plane $\phi = 0$ this assumption allows us to set

$$\beta_m = 0 \quad (0 < m \leqslant l). \tag{2.13}$$

For reasons of mathematical convenience we also introduce the normalization condition

$$\langle |w_l|^2 \rangle = \sum_{m=0}^{l} \alpha_m^2 = 1. \tag{2.14}$$

Before the higher orders in ϵ of the problem (2.4), (2.5) can be studied the solution of the adjoint problem to the linear homogeneous problem (2.8), (2.5) must be discussed. The adjoint operators $\mathbf{W}^+$ and $\mathbf{U}^+$ are given by the transposed forms $\mathbf{W}^{\mathrm{T}}$, $\mathbf{U}^{\mathrm{T}}$ of the operators $\mathbf{W}$ and $\mathbf{U}$. A complete set of independent solutions that are symmetric with respect to $\phi = 0$ can be written in the form

$$\mathbf{X}_i^+ = \begin{pmatrix} f^+(r) \\ g^+(r) \end{pmatrix} w_l^{(i)} \quad (i = 0, 1, \ldots, l), \tag{2.15a}$$

where

$$w_l^{(i)} = \left\{ \begin{array}{ll} w_l = \sum_{m=0}^{l} \alpha_m Y_l^m & (i = 0), \\ & \\ \alpha_{i+1} Y_l^i - \alpha_i Y_l^{i+1} & (0 < i < l), \\ \alpha_0 Y_l^l - \alpha_l Y_l^0 & (i = l). \end{array} \right\} \tag{2.15b}$$

The special choice of the functions (2.15*b*) allows the condition

$$\langle w_l^{(i)} w_l \rangle = \delta_{i0} \tag{2.16}$$

to be satisfied (where δ_{ik} is the Kronecker symbol).

Of particular interest is the self-adjoint case, which is characterized by the condition

$$\left(\frac{r\hat{\gamma}(r)}{T'(r)} \right)' = 0 \tag{2.17}$$

for all commonly used boundary conditions of the form (2.5). In the self-adjoint case

$$f^+(r) \propto f(r), \quad g^+(r) \propto g(r) \tag{2.18}$$

holds, and separate computations of the adjoint linear problem are not required.

3. Perturbation analysis

To order ϵ^2 (2.4) yields

$$(\mathbf{W}+R^{(0)}\mathbf{U}).\mathbf{X}^{(2)} = \mathbf{Q}(\mathbf{X}^{(1)},\mathbf{X}^{(1)})-R^{(1)}\mathbf{U}.\mathbf{X}^{(1)}. \tag{3.1}$$

A solution of the inhomogeneous equation (3.1) exists if the right-hand side of (3.1) is orthogonal to all solutions of the adjoint homogeneous problem. Since the right-hand side is a symmetric function in ϕ, it suffices to multiply it by the set of solutions (2.15) and average the result over the fluid layer. The resulting equations can be written in the form

$$R^{(1)} = M_0(l)\langle w_l w_l w_l\rangle, \tag{3.2a}$$

$$0 = \langle w_l^{(i)} w_l w_l\rangle \quad (i = 1, \ldots, l), \tag{3.2b}$$

where the property (B1) (derived in appendix B) of spherical harmonics has been used. The function $M_0(l)$ is given by

$$M_0(l) = \left\langle \frac{fg'g^+}{r}+\frac{gg^+(f'+f/r)}{2r}+\frac{f^+}{P}\left[(fr)''D_l f'+\frac{l(l+1)(fr)'D_l f}{4r^2}-\tfrac{1}{2}r[(D_l f)^2]'\right]\right\rangle$$
$$\times\langle(r_0+\tfrac{1}{2})\hat{\gamma}(r)f^+g\rangle^{-1}, \tag{3.3}$$

where the operator D_l is defined by

$$D_l \equiv \frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{l(l+1)}{r^2}. \tag{3.4}$$

It is easily seen that $M_0(l) = 0$ in the self-adjoint case because of the property (2.18). Because of the symmetry property of spherical harmonics the relationship

$$w_l(\theta,\phi) = (-1)^l w_l(\pi-\theta,\phi+\pi)$$

holds, with the consequence that the right-hand sides of (3.2a, b) vanish whenever l is an odd integer. We thus arrive at the conclusions that $R^{(1)}$ always vanishes for odd l, and that it vanishes for all l in the case of a self-adjoint linear problem (2.4), (2.6). The first of these conclusions has been demonstrated in B75, and the second agrees with the validity of the energy principle in the self-adjoint case (Joseph & Carmi 1966).

Since (3.2a, b) have been discussed in detail in B75, we proceed by deriving the solution of (3.1). The structure of the latter equation permits us to write

$$\mathbf{X}^{(2)} \equiv \begin{pmatrix}\Phi^{(2)}(r_0+\frac{1}{2})\\ \Theta^{(2)}\end{pmatrix} \equiv \sum_{k=0}^{l}\begin{pmatrix}F_k(r)\\ G_k(r)\end{pmatrix}\sum_{j=0}^{2k}\langle Y_{2k}^j w_l w_l\rangle Y_{2k}^j, \tag{3.5}$$

i.e. the solution is obtained by expanding the right-hand side of (3.1) in terms of spherical harmonics $Y_p^i(\theta,\phi)$, of which only those of even degree are needed because of the symmetry of the problem. Since the solution (3.5) is determined only up to an arbitrary contribution of the solution of the homogeneous equation, a normalization condition is needed. For mathematical convenience we choose the condition

$$\langle \mathbf{X}_0^+.\mathbf{U}.\mathbf{X}^{(n)}\rangle = 0 \quad (n = 2,3\ldots). \tag{3.6}$$

To order ϵ^3 (2.4) yields

$$(\mathbf{W}+R^{(0)}\mathbf{U}).\mathbf{X}^{(3)} = \mathbf{Q}(\mathbf{X}^{(1)},\mathbf{X}^{(2)})+\mathbf{Q}(\mathbf{X}^{(2)},\mathbf{X}^{(1)})-R^{(2)}\mathbf{U}.\mathbf{X}^{(1)}-R^{(1)}\mathbf{U}.\mathbf{X}^{(2)}. \tag{3.7}$$

Multiplication of the right-hand side by the set of solutions (2.15) and averaging over the fluid layer yields

$$R^{(2)} = \sum_{k=0}^{l} M(l,k) \sum_{j=0}^{2k} \langle Y_{2k}^{j} w_l w_l \rangle^2, \tag{3.8a}$$

$$0 = \sum_{k=0}^{l} M(l,k) \sum_{j=0}^{2k} \langle Y_{2k}^{j} w_l w_l \rangle \langle Y_{2k}^{j} w_l w_l^{(i)} \rangle \quad (i = 1, \ldots, l), \tag{3.8b}$$

where $M(l, k)$ is given by

$$M(l,k) \equiv \Big\langle g^{+}\Big[\frac{fG_k'}{r} + \frac{k(2k+1)}{l(l+1)}\frac{G_k(f'+f/r)}{r} + \frac{k(2k+1)}{l(l+1)\,r}\Big(2g'F_k + g\Big(F_k' + \frac{F_k}{r}\Big)\Big)\Big]$$
$$+\frac{f^{+}}{P}\Big[(rf)''(D_kF_k)' + \frac{k(2k+1)}{r^2}\Big((rf)'D_kF_k + (rF_k)'D_lf\Big)$$
$$-r(D_kF_kD_lf)' + (rF_k)''(D_lf)'\Big]\Big\rangle \langle (r_0+\tfrac{1}{2})\hat{\gamma}(r)f^{+}g\rangle^{-1}. \tag{3.9}$$

The investigation of the nonlinear equations (3.8a, b) is facilitated by the apparent validity of the relationship

$$\sum_{p=0}^{2k} \langle Y_{2k}^{p} w_l w_l \rangle \langle Y_{2k}^{p} w_l w_l^{(i)} \rangle = C(l,k) \sum_{p=0}^{2} \langle Y_{2}^{p} w_l w_l \rangle \langle Y_{2}^{p} w_l w_l^{(i)} \rangle \tag{3.10}$$

for $k \geqslant 1$ and for $i = 1, \ldots, l$, where $C(l, k)$ is a function of l and k only. For all values of l that have been investigated this relationship was found to be valid. But so far it has not been possible to prove its general validity. When the relationship (3.10) is accepted as a hypothesis, (3.8b) can be rewritten in the form

$$0 = \sum_{p=0}^{2} \langle Y_{2}^{p} w_l w_l \rangle \langle Y_{2}^{p} w_l w_l^{(i)} \rangle \quad (i = 1, \ldots, l), \tag{3.11}$$

provided that the common factor in all the equations (3.8b) does not vanish:

$$\sum_{k=1}^{l} C(l,k)\, M(l,k) \neq 0. \tag{3.12}$$

Since this condition is generally satisfied, the system of $l+1$ equations for the $l+1$ unknowns α_m given by (2.14) and (3.11) becomes independent of the r-dependence of the problem. This remarkable fact extends to cases of odd l and to the self-adjoint case the property that the possible steady solutions of the form (2.10) are independent of the physical specification of the problem. It permits the determination of possible patterns of convection without knowledge of the functions $f(r)$, $g(r)$, $F_k(r)$, $G_k(r)$, which often require cumbersome computations. Before investigating (3.11) for special values of l we consider the general problem of stability.

4. Stability analysis

In general, several solutions of the form (2.10) satisfy the solvability conditions (3.2) and (3.8). It thus becomes necessary to distinguish the physically realizable solution among the manifold of existing solutions by its stability property. The stability problem described by (2.6) and the corresponding boundary conditions (2.5) must be solved for all possible disturbances $\tilde{\mathbf{X}}$. The steady solution $\mathbf{X}$ is stable if the real parts of all eigenvalues σ are negative or zero; otherwise it is unstable. After the expressions (2.7) are inserted in (2.6) it becomes obvious that the solution $\tilde{\mathbf{X}}$, σ of the linear homogeneous problem (2.6) can be obtained in the form

$$\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^{(1)} + \epsilon\tilde{\mathbf{X}}^{(2)} + \epsilon^2\tilde{\mathbf{X}}^{(3)} + \ldots, \quad \sigma = \sigma^{(0)} + \epsilon\sigma^{(1)} + \epsilon^2\sigma^{(2)} + \ldots. \tag{4.1}$$

To order ϵ^0 the equation

$$(\mathbf{W}+R^{(0)}\mathbf{U})\tilde{\mathbf{X}}^{(1)} = \sigma^{(0)}\mathbf{V}\tilde{\mathbf{X}}^{(1)} \tag{4.2}$$

is obtained. Since $R^{(0)}$ is assumed to be the minimum value for which (2.8) can be satisfied, $\sigma^{(0)} = 0$ represents the growth rate with maximal real part. Accordingly $\tilde{\mathbf{X}}^{(1)}$ can be written in the form

$$\tilde{\mathbf{X}}^{(1)} = \begin{pmatrix} f(r) \\ g(r) \end{pmatrix} \tilde{w}_l(\theta,\phi), \tag{4.3}$$

where

$$\tilde{w}_l(\theta,\phi) \equiv \sum_{m=0}^{l} (\tilde{\alpha}_m \cos m\phi + \tilde{\beta}_m \sin m\phi)\, P_l^m(\cos\theta) \equiv \sum_{m=0}^{l} (\tilde{\alpha}_m Y_l^m + \tilde{\beta}_m \hat{Y}_l^m). \tag{4.4}$$

The special choice (2.15) of the solutions of the adjoint homogeneous problem does not provide any advantage in the discussion of the solvability conditions of the stability equations. Thus the following complete set of solutions of the adjoint problem will be used.

$$\mathbf{X}_p^+ = \begin{pmatrix} f^+(r) \\ g^+(r) \end{pmatrix} \left\{ \begin{matrix} Y_l^p(\theta,\phi) & (p = 0,1,\ldots,l) \\ \hat{Y}_l^{-p}(\theta,\phi) & (p = -l,\ldots,-1) \end{matrix} \right\}. \tag{4.5}$$

To order ϵ^1 in (2.6) the equation

$$(\mathbf{W}+R^{(0)}\mathbf{U}).\tilde{\mathbf{X}}^{(2)} = \sigma^{(1)}\mathbf{V}.\tilde{\mathbf{X}}^{(1)} - R^{(1)}\mathbf{U}.\tilde{\mathbf{X}}^{(1)} + \mathbf{Q}(\mathbf{X}^{(1)},\tilde{\mathbf{X}}^{(1)}) + \mathbf{Q}(\tilde{\mathbf{X}}^{(1)},\mathbf{X}^{(1)}) \tag{4.6}$$

is obtained. Multiplication of the right-hand side by the set of functions (4.5) and averaging over the fluid layer yields

$$(R^{(1)} - \sigma^{(1)} M_2(l))\,\tilde{\alpha}_i = 2M_0(l)\langle Y_l^i w_l \tilde{w}_l\rangle \quad (i = 0,1,\ldots,l), \tag{4.7a}$$

$$(R^{(1)} - \sigma^{(1)} M_2(l))\,\tilde{\beta}_i = 2M_0(l)\langle \hat{Y}_l^i w_l w_l\rangle \quad (i = 1,2,\ldots,l), \tag{4.7b}$$

where the positive expression $M_2(l)$ is defined by

$$M_2(l) \equiv \langle gg^+ - P^{-1}l(l+1)f^+D_lf\rangle \langle l(l+1)\hat{\gamma}(r)f^+g\rangle^{-1}. \tag{4.8}$$

As in the discussion of the conditions (3.2a, b) it is readily seen that $\sigma^{(1)}$ vanishes in the case of odd integers l and in the self-adjoint case. In the other case the eigenvalues $\sigma^{(1)}$ are determined by the condition that the determinant of the coefficient matrix for the unknowns $\tilde{\alpha}_i$, $\tilde{\beta}_i$ vanishes. A particular solution of (4.7) is given by

$$M_2\sigma_1^{(1)} = -R^{(1)}, \quad \tilde{\alpha}_i = \alpha_i, \quad \tilde{\beta}_i = 0 \quad (i = 0,1,\ldots,l), \tag{4.9}$$

which indicates that the solution with $\epsilon R^{(1)} < 0$ is unstable.

In cases of odd l or when the problem is nearly self-adjoint the contribution $\sigma^{(2)}$ to the growth rate σ becomes important in deciding the stability of the steady solutions. Assuming that $\sigma^{(1)}$ and $R^{(1)}$ are vanishing (or at least small compared with the other terms on the right-hand side of (4.6)) we obtain as a solution of (4.6)

$$\tilde{\mathbf{X}}^{(2)} \equiv \begin{pmatrix} \tilde{\mathbf{\Phi}}^{(2)}(r_0+\frac{1}{2}) \\ \tilde{\mathbf{\Theta}}^{(2)} \end{pmatrix} = 2\sum_{k=0}^{l} \begin{pmatrix} F_k(r) \\ G_k(r) \end{pmatrix} \sum_{j=0}^{2k} (\langle w_l\tilde{w}_l Y_{2k}^j\rangle Y_{2k}^j + \langle w_l\tilde{w}_l \hat{Y}_{2k}^j\rangle \hat{Y}_{2k}^j). \tag{4.10}$$

Using this expression, the solvability conditions for the equation of order ϵ^2,

$$(\mathbf{W}+R^{(0)}\mathbf{U}).\tilde{\mathbf{X}}^{(3)} = \sigma^{(2)}\mathbf{V}.\mathbf{X}^{(1)} - R^{(2)}\mathbf{U}.\tilde{\mathbf{X}}^{(1)} + \mathbf{Q}(\tilde{\mathbf{X}}^{(2)},\mathbf{X}^{(1)}) + \mathbf{Q}(\tilde{\mathbf{X}}^{(1)},\mathbf{X}^{(2)})$$
$$+\mathbf{Q}(\mathbf{X}^{(2)},\tilde{\mathbf{X}}^{(1)} + \mathbf{Q}(\mathbf{X}^{(1)},\tilde{\mathbf{X}}^{(2)}), \tag{4.11}$$

can be written in the form

$$(R^{(2)}-\sigma^{(2)}M_2(l))\tilde{\alpha}_i = \sum_{k=0}^{l} M(l,k) \sum_{j=0}^{2k} (2\langle w_l \tilde{w}_l Y_{2k}^j\rangle \langle w_l Y_{2k}^j Y_l^i\rangle + \langle w_l w_l Y_{2k}^j\rangle \langle \tilde{w}_l Y_{2k}^j Y_l^i\rangle) \quad (i = 0, 1, \ldots, l), \tag{4.12a}$$

$$(R^{(2)}-\sigma^{(2)}M_2(l))\tilde{\beta}_i = \sum_{k=0}^{l} M(l,k) \sum_{j=0}^{2k} (2\langle w_l \tilde{w}_l \hat{Y}_{2k}^j\rangle \langle w_l \hat{Y}_{2k}^j \hat{Y}_l^i\rangle + \langle w_l w_l Y_{2k}^j\rangle \langle \tilde{w}_l Y_{2k}^j Y_l^i\rangle) \quad (i = 1, 2, \ldots, l). \tag{4.12b}$$

The system (4.12a, b) of $2l+1$ linear homogeneous equations is solvable if and only if the determinant of the matrix of the coefficients of the unknowns $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ vanishes. The eigenvalues $\sigma^{(2)}$ can thus be determined as the zeros of the determinant. Unlike the case of the solvability conditions (3.8) for the steady solution, it is not possible to evaluate the stability equations without detailed knowledge of the radial dependence of the problem. The main general conclusion to be drawn from (4.12a, b) is that the two equations are not coupled and that the matrices of coefficients are symmetric. The eigenvalues $\sigma^{(2)}$ are thus real and can be determined by considering (4.12a, b) separately.

A few eigenvalues $\sigma^{(2)}$ can be determined directly. By comparing (4.11) with (3.7) it can be seen that

$$\tilde{\mathbf{X}}^{(1)} = \mathbf{X}^{(1)} \quad \text{or} \quad \tilde{\alpha}_i = \alpha_i \quad (i = 0, \ldots, l)$$

is a solution of (4.12a) corresponding to

$$M_2 \sigma_1^{(2)} = -2R^{(2)}.$$

In addition it is known that there are two independent possibilities of rotating the steady solution on the sphere. Therefore there must be at least two eigenvalues $\sigma^{(2)} = 0$.

5. Steady solutions

In this section the remarkable property will be exploited that the coefficients α_i are independent of the radial dependence of the problem according to the relationship (3.10). The attention will be focused on cases for which $R^{(1)}$ vanishes, since the problem has been treated for $R^{(1)} \neq 0$ in B75. Although the relationship (3.10) permits a general solution for the coefficients α_i, it does not simplify the mathematical structure of the equations (3.8). Because of the highly nonlinear form of (3.11) only cases of low values of l can be solved explicitly. It is readily seen that the axisymmetric solution $\alpha_0^2 = 1, \alpha_i = 0$ $(i = 1, \ldots, l)$ represents a solution of the system (3.8) for all values of l because (3.8b) vanish identically. But there does not seem to exist any other solution that can be defined independently of l. Each l must thus be discussed separately.

5.1. *The case* $l = 1$

Equations (2.14) and (3.8) yield for $l = 1$

$$R^{(2)} = M(1,0)+\tfrac{4}{5}M(1,1), \tag{5.1a}$$

$$\alpha_1^2 = 1-\alpha_0^2. \tag{5.1b}$$

The possible choices of coefficients admitted by (5.1b) correspond to all possible inclinations of the axisymmetric solution $\alpha_0^2 = 1$ with respect to the axis of the co-ordinate system within the plane $\phi = 0$. The solvability conditions (3.8) do not constrain the manifold of solutions (2.10) because each one of the latter represents a transformation of the axisymmetric solution in the case $l = 1$. Thus the stability problem becomes trivial in this case because of the absence of competing states of convection.

5.2. *The case* $l = 2$

Unlike the case $l = 1$ a true pattern degeneracy exists for $l = 2$. The term pattern degeneracy applies to physically distinguishable solutions, in contrast to the orientational degeneracy, which originates from the rotations of a given solution on the surface of the sphere. In the non-self-adjoint case the axisymmetric mode alone satisfies the solvability conditions (3.11) according to B75. But in the self-adjoint case the pattern degeneracy cannot be removed since (3.11) vanish identically for $l = 2$. Since the self-adjoint case represents a relatively special problem and since a detailed discussion is given by Golubitsky & Schaeffer (1982) we shall not consider this case further. The expression for $R^{(2)}$ is given by

$$R^{(2)} = M(2,0) + \tfrac{20}{49}M(2,1) + \tfrac{36}{49}M(2,2), \tag{5.2}$$

independently of the values of the coefficients α_i.

5.3. *The case* $l = 3$

Equations (3.11) assume the following form for $l = 3$:

$$\tfrac{1}{5}\alpha_1\alpha_2(-8\alpha_0^2 + 2\alpha_1^2 + 5\alpha_2^2 - 10\alpha_3^2) + 2\alpha_0\alpha_3(\alpha_2^2 - \alpha_1^2) \\ + 15^{-\frac{1}{2}}\{4\alpha_1\alpha_2(-\alpha_0\alpha_2 - 4\alpha_1\alpha_3) + \tfrac{1}{3}\alpha_2^3\alpha_3 + 2\alpha_1^3\alpha_0\} = 0, \tag{5.3a}$$

$$\tfrac{1}{3}\alpha_2\alpha_3(12\alpha_0^2 + 4\alpha_1^2 - 5\alpha_2^2) + 2\alpha_0\alpha_1(\alpha_3^2 - \alpha_2^2) \\ + 15^{-\frac{1}{2}}\{2\alpha_1^2(\alpha_2\alpha_1 - \alpha_3\alpha_0) + \alpha_1\alpha_2 5(2\alpha_3^2 - \alpha_2^2)\} = 0, \tag{5.3b}$$

$$\tfrac{1}{5}\alpha_3\alpha_0(-12\alpha_0^2 - 7\alpha_1^2 - 5\alpha_2^2 + 15\alpha_3^2) + 2\alpha_1\alpha_2(\alpha_0^2 - \alpha_3^2) \\ + 15^{-\frac{1}{2}}\{\alpha_1\alpha_2(2\alpha_1\alpha_3 + 5\alpha_2\alpha_0) - 2\alpha_1^3\alpha_0\} = 0. \tag{5.3c}$$

Three distinct solutions of (5.3*a–c*) have been found, which are listed here in the form in which a maximum number of the coefficients α_i vanish. In addition there exists an infinite manifold of solutions which are obtained from those three solutions by moving the polar axis within the plane $\phi = 0$ with respect to a fixed axis.

Solution A is the axisymmetric solution

$$\alpha_0^2 = 1, \quad \alpha_i = 0 \quad (i = 1, 2, 3), \tag{5.4a}$$

yielding
$$R^{(2)} = M(3,0) + \tfrac{16}{45}M(3,1) + \tfrac{36}{121}M(3,2) + \tfrac{10000}{14157}M(3,3). \tag{5.4b}$$

Solution B exhibits a tetrahedral symmetry

$$\alpha_2^2 = 1, \quad \alpha_0 = \alpha_1 = \alpha_3 = 0, \tag{5.5a}$$

corresponding to
$$R^{(2)} = M(3,0) + \tfrac{84}{121}M(3,2) + \tfrac{800}{1573}M(3,3). \tag{5.5b}$$

Some other simple descriptions of the tetrahedral solution are given by

$$\alpha_1^2 = \tfrac{5}{8}, \quad \alpha_3^2 = \tfrac{3}{8}, \quad \alpha_0 = \alpha_2 = 0, \quad \alpha_1\alpha_3 > 0, \tag{5.5c}$$

or
$$\alpha_0^2 = \tfrac{5}{9}, \quad \alpha_3^2 = \tfrac{4}{9}, \quad \alpha_1 = \alpha_2 = 0. \tag{5.5d}$$

Solution C is given by
$$\alpha_3^2 = 1, \quad \alpha_0 = \alpha_1 = \alpha_2 = 0, \tag{5.6a}$$

and yields
$$R^{(2)} = M(3,0) + \tfrac{5}{9}M(3,1) + \tfrac{9}{121}M(3,2) + \tfrac{11575}{14157}M(3,3). \tag{5.6b}$$

The same solution can also be given in the transformed positions

$$\alpha_1^2 = \tfrac{15}{16}, \quad \alpha_3^2 = \tfrac{1}{16}, \quad \alpha_0 = \alpha_2 = 0, \quad \alpha_1\alpha_3 < 0,$$

or

$$\alpha_0^2 = \tfrac{5}{8}, \quad \alpha_2^2 = \tfrac{3}{8}, \quad \alpha_1 = \alpha_3 = 0.$$

It is expected that there does not exist any solution of equations (5.3), (2.14) that does not represent a transformation of solutions A, B or C. But we have not been able to produce a rigorous proof of this hypothesis.

6. Stable convection flows

The existence of more than one steady solution for $l \geqslant 2$ makes it necessary to investigate the stability of the solutions in order to determine the physically preferred mode of convection. In contrast to (3.11) determining the steady solutions, the equations determining the disturbances and their growth rates depend in general on the radial dependence of the problem. A special case must be chosen in order to derive explicit expressions for the eigenvalues σ. In §7 such a case is considered which permits an analytical evaluation of all functions of the radial co-ordinate. Before entering that analysis it is useful to derive general expressions for the eigenvalues $\sigma^{(2)}$ in the cases $l = 1, 2$ and 3. As will be shown in this section the question of stability can be decided in many instances without reference to the numerical values of special cases.

6.1. *The case $l = 1$*

The analysis of the stability equations (4.12) is particularly simple in the case $l = 1$. The vanishing of the determinant of the matrix of coefficients of the unknown α_i in the system of equations (4.12a) yields

$$\det\begin{pmatrix} M_2\sigma^{(2)}/R^{(2)} + 2\alpha_0^2 & 2\alpha_0\alpha_1 \\ 2\alpha_0\alpha_1 & M_2\sigma^{(2)}/R^{(2)} + 2\alpha_1^2 \end{pmatrix} = 0,$$

with

$$\sigma_1^{(2)} = 0, \quad \sigma_2^{(2)} = -2R^{(2)}/M_2 \tag{6.1a}$$

as eigenvalues. Equation (4.12b) yields a third eigenvalue

$$\sigma_3^{(2)} = 0. \tag{6.1b}$$

These results indicate that the steady solution in the case $l = 1$ is stable as expected. The vanishing eigenvalues $\sigma^{(2)}$ correspond to two independent rotations on the sphere, while the negative eigenvalue corresponds to a disturbance of the same form as the steady solution. The case $l = 1$ has also been discussed by Geiger (1977) in the context of convection in a full sphere.

6.2. *The case $l = 2$*

The determinant for (4.12a) can be written in the form

$$\det\begin{pmatrix} M_2\sigma^{(2)}/R^{(2)} + 2\alpha_0^2 & 2\alpha_0\alpha_1 & 2\alpha_0\alpha_2 \\ 2\alpha_1\alpha_0 & M_2\sigma^{(2)}/R^{(2)} + 2\alpha_1^2 & 2\alpha_1\alpha_2 \\ 2\alpha_0\alpha_2 & 2\alpha_1\alpha_2 & M_2\sigma^{(2)}/R^{(2)} + 2\alpha_2^2 \end{pmatrix} = 0,$$

which yields the eigenvalues

$$\sigma^{(2)} = -2R^{(2)}/M_2, \quad \sigma_2^{(2)} = \sigma_3^{(2)} = 0. \tag{6.2a}$$

Equations (4.12b) yield the additional eigenvalues

$$\sigma_4^{(2)} = \sigma_5^{(2)} = 0. \tag{6.2b}$$

These results are independent of the coefficients α_i of the steady solution. The stability analysis thus does not allow different solutions to be distinguished to this order of the problem. Higher orders must be investigated in the self-adjoint case to resolve the degeneracy. But in the general non-self-adjoint case the problem is essentially solved to order ϵ^2. According to B75 the solutions $\alpha_0 = \pm 1$ are the only ones satisfying the solvability conditions at that order. The growth rates for both solutions can be written in the form

$$M_2\sigma_1 = -\epsilon R^{(1)} - 2\epsilon^2 R^{(2)}, \tag{6.3a}$$

$$M_2\sigma_2 = M_2\sigma_4 = 0, \tag{6.3b}$$

$$M_2\sigma_3 = 3\epsilon R^{(1)} = M_2\sigma_5, \tag{6.3c}$$

where

$$R^{(1)} = \tfrac{1}{7}20^{\frac{1}{2}} M_0(2)\alpha_0$$

may have both signs, while $R^{(2)}$ is given by expression (5.2). Assuming that the latter is positive, as is usually the case, we find that the stability of the axisymmetric solution requires

$$\alpha_0 M_0(2) < 0. \tag{6.4}$$

When this condition is satisfied the respective axisymmetric solution is stable for

$$\epsilon \geqslant |R^{(1)}|/2R^{(2)}. \tag{6.5}$$

Here, as elsewhere in the paper, it has been presumed that ϵ assumes positive values only. The condition (6.5) indicates that the axisymmetric solution is stable on its upper branch in the (ϵ, R)-diagram, since the equality sign in (6.5) corresponds to the point where the Rayleigh number reaches its minimum value. In this respect the axisymmetric solution resembles closely the hexagon solution for a plane convection layer (Busse 1967). As in the latter case the approximate validity of conditions such as (6.5) requires, of course, that $|R^{(1)}| \ll R^{(2)}$, since otherwise higher-order terms must be taken into account.

6.3. *The case $l = 3$*

In general it cannot be expected that the stability equations (4.12*a*, *b*) permit explicit analytical expressions for the eigenvalues $\sigma^{(2)}$. But in the cases of the steady solutions (5.4*a*), (5.5*a*), (5.6*a*) the determinants of the matrices from which the eigenvalues $\sigma^{(2)}$ are derived become especially simple. In cases A and C the non-diagonal elements of the coefficient matrices of (4.12*a*, *b*) vanish entirely, and in case B only two non-diagonal elements differ from zero. Moreover those latter elements are proportional to the corresponding diagonal elements such that $\sigma^{(2)} = 0$ is an eigenvalue. The following expressions for the seven eigenvalues for each of the solutions are found.

Case A, $\alpha_0^2 = 1$:

$$\left.\begin{aligned} M_2\sigma_1^{(2)} &= -2R_A^{(2)}, \\ M_2\sigma_2^{(2)} &= M_2\sigma_5^{(2)} = 0, \\ M_2\sigma_3^{(2)} &= -\tfrac{8}{21}M(3,1) + \tfrac{72}{121}M(3,2) - \tfrac{1400}{121\cdot 39}M(3,3) = M_2\sigma_6^{(2)}, \\ M_2\sigma_4^{(2)} &= \tfrac{4}{5}M(3,1) - \tfrac{108}{121}M(3,2) + \tfrac{700}{121\cdot 13}M(3,3) = M_2\sigma_7^{(2)}. \end{aligned}\right\} \tag{6.6}$$

Case B, $\alpha_2^2 = 1$:

$$\left.\begin{aligned} M_2\sigma_1^{(2)} &= -2R_B^{(2)}, \\ M_2\sigma_2^{(2)} &= -\tfrac{8}{9}M(3,1) + \tfrac{120}{121}M(3,2) - \tfrac{7000}{117\cdot 121}M(3,3), \\ M_2\sigma_3^{(2)} &= -\tfrac{8}{9}M(3,1) + \tfrac{120}{121}M(3,2) - \tfrac{400}{9\cdot 121}M(3,3) = M_2\sigma_5^{(2)}, \\ M_2\sigma_4^{(2)} &= M_2\sigma_6^{(2)} = M_2\sigma_7^{(2)} = 0. \end{aligned}\right\} \tag{6.7}$$

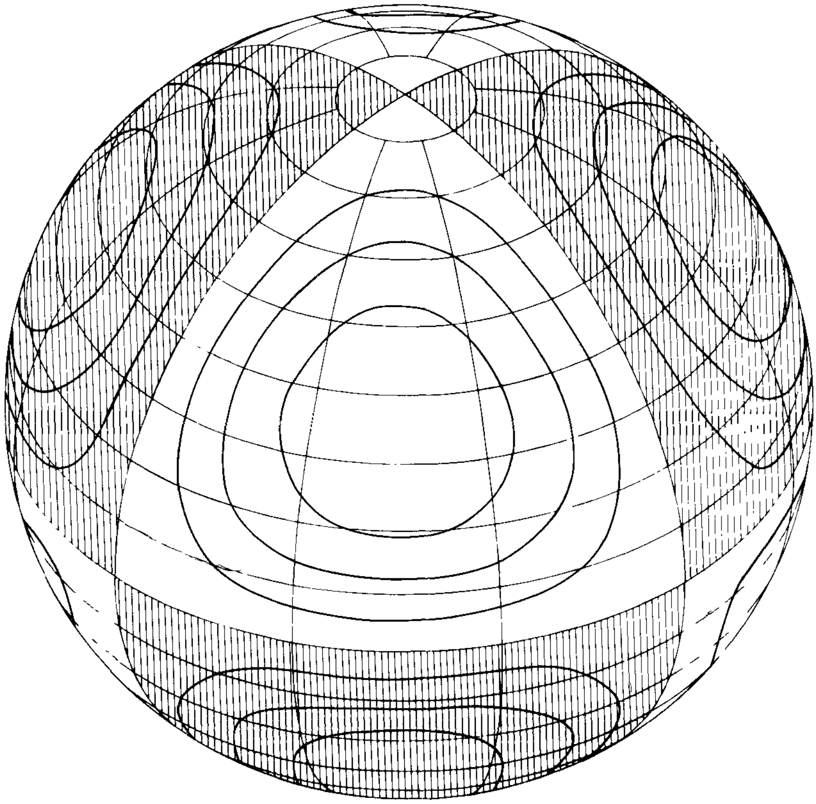

FIGURE 1. Lines of constant radial velocity corresponding to quarters of the maximum value. Motion is inward in the shaded areas and outward in the white areas or *vice versa*.

Case C, $\alpha_3^2 = 1$:

$$\left.\begin{aligned} M_2\sigma_1^{(2)} &= -2R_C^{(2)}, \\ M_2\sigma_2^{(2)} &= M(3,1)-\tfrac{135}{121}M(3,2)+\tfrac{857}{13\cdot 121}M(3,3), \\ M_2\sigma_3^{(2)} &= \tfrac{2}{3}M(3,1)-\tfrac{90}{121}M(3,2)+\tfrac{1750}{99\cdot 13}M(3,3) = M_2\sigma_5^{(2)}, \\ M_2\sigma_4^{(2)} &= M_2\sigma_6^{(2)} = M_2\sigma_7^{(2)} = 0. \end{aligned}\right\} \tag{6.8}$$

These expressions are as general as the expressions (5.4*b*), (5.5*b*), (5.6*b*) for $R^{(2)}$. While it can generally be concluded that $R^{(2)}$ is positive because of the dominating positive contribution $M(3,0)$, little can be concluded about the signs of the above expressions for $\sigma^{(2)}$. But the remarkable property that

$$\sigma_3^{(2)} = -\tfrac{2}{3}\sigma_4^{(2)} \tag{6.9}$$

holds in case A permits the immediate conclusion that the axisymmetric solution is always unstable except in the singular case $\sigma_3^{(2)} = 0$, which requires consideration of contributions of higher order. If $M(3,3)$ is positive and $\sigma_3^{(2)}$ of case B negative, the tetrahedral solution is stable because $\sigma_3^{(2)} > \sigma_2^{(2)}$, while solution C is unstable because of the relationship

$$M_2(\sigma_{3B}^{(2)}+\tfrac{8}{9}\sigma_{2C}^{(2)}) = \tfrac{184}{1573}M(3,3), \tag{6.10}$$

where the second subscript indicates the respective steady solution. As will be discussed in more detail in §7, the above assumptions are usually satisfied.

Since in addition to (6.10) the relationship

$$M_2(\sigma_{3B}^{(2)}+\tfrac{4}{3}\sigma_{3C}^{(2)}) = -\tfrac{200}{99\cdot 429}M(3,3) \tag{6.11}$$

holds, solution C is unstable if solution B is stable, and *vice versa*, independent of the sign of $M(3,3)$. Since at least one solution must be stable according to the extremum principle given in appendix C, it can be concluded that one and only one solution is physically realizable. A sketch of solution B displaying the tetrahedral symmetry is shown in figure 1.

7. Convection in a thin spherical fluid layer

In order to discuss in more detail the expressions derived in the preceding sections, a special case will be considered in this section. Because it permits simple analytical expressions for the radial dependence of all dependent variables, the limit of a thin fluid shell with nearly insulating boundaries has been selected. Other cases of more physical interest are considered in Riahi, Geiger & Busse (1982).

We consider (2.4) in the special case

$$\hat{\gamma}(r) = 1, \quad T'(r) = r(r_0+\tfrac{1}{2})^{-1}, \tag{7.1}$$

and assume rigid boundaries of uniform thermal conductivity on both sides of the fluid layer:

$$\Phi = \frac{\partial}{\partial r}\Phi = 0 \quad \text{at} \quad r = r_0, \quad r_0+1, \tag{7.2a}$$

$$\frac{\partial}{\partial r}\Theta = \beta\frac{\partial}{\partial r}\Theta_e, \quad \Theta = \Theta_e \quad \text{at} \quad r = r_0, \quad r_0+1. \tag{7.2b}$$

Θ_e denotes the temperature perturbation outside the fluid layer and β is the ratio of the thermal conductivity outside to that inside the fluid layer. Following the corresponding analysis for a plane layer (Busse & Riahi 1980; hereinafter referred to as BR) we look for solutions of the problem in the limit of small β and introduce

$$\gamma \equiv \beta^{\frac{2}{3}} \tag{7.3}$$

as perturbation parameter. At the same time the limit of a thin shell is assumed such that r_0 tends to infinity with $\beta^{-\frac{1}{3}}$:

$$(r_0+\tfrac{1}{2})^{-2} = \eta^2\gamma, \tag{7.4}$$

where η is a parameter of the order unity. Solutions of the problem described by (2.4), (7.1), (7.2) can be obtained by expanding the coefficients in the series (2.7) in terms of powers of γ:

$$\mathbf{X}^{(n)} = \sum_{\mu=0}^{\infty}\gamma^{\mu}\mathbf{X}_{\mu}^{(n)}, \quad R^{(n)} = \sum_{\mu=0}^{\infty}\gamma^{\mu}R_{\mu}^{(n)} \quad (n = 0, 1, \ldots). \tag{7.5}$$

Starting with the linear problem given by the terms of order ϵ in (2.4) it is found that the results are analogous to those of BR. After introducing

$$\left.\begin{aligned} f(r) &= f_0(r)+\gamma f_1(r)+\ldots, \\ g(r) &= g_0(r)+\gamma g_1(r)+\ldots, \end{aligned}\right\} \tag{7.6}$$

and using $z \equiv r-(r_0+\frac{1}{2})$ as new variable, the results

$$g_0(z) = c, \quad f_0(z) = cR_0^{(0)}(z^2-\tfrac{1}{4})^2/4!, \quad R_0^{(0)} = 720 \tag{7.7}$$

are found at lowest order, where c is a constant that for simplicity will be set equal to unity. The exterior temperature distribution Θ_e enters the problem only in the solvability condition for the order γ^2 of the temperature equation. Using the equation $\nabla^2\Theta_e = 0$ outside the fluid layer, the boundary condition (7.2*b*) can be rewritten as

$$\frac{\partial}{\partial z} g(z) = \frac{-\gamma^2\eta(l+1)\,g(z)\,(r_0+\frac{1}{2})}{r_0+1} \quad (z = +\tfrac{1}{2}), \tag{7.8a}$$

$$\frac{\partial}{\partial z} g(z) = \frac{\gamma^2\eta l g(z)\,(r_0+\frac{1}{2})}{r_0} \quad (z = -\tfrac{1}{2}). \tag{7.8b}$$

At second order in the linear problem the results

$$g_1(z) = l(l+1)\,\eta^2(\tfrac{31}{21}(\tfrac{1}{2})^6 - \tfrac{7}{18}z^2 + \tfrac{5}{4}z^4 - z^6), \tag{7.9a}$$

$$\begin{aligned} f_1(z) = {} & l(l+1)\,\eta^2 R_0^{(0)}[-(2z)^{10} + 15(2z)^8 + 126(2z)^6 \\ & -810(2z)^4 + 1187(2z)^2 - 517]\,6!/10!2^{10}, \end{aligned} \tag{7.9b}$$

$$R_1^{(0)} = 720\left[\frac{2l+1}{\eta l(l+1)} + \frac{17\eta^2}{462} l(l+1)\right] \tag{7.9c}$$

are obtained.

The minimum of the expression (7.9*c*) as a function of l is achieved by

$$\begin{aligned} l &= 1 \quad (\eta \geqslant \tfrac{154}{51}), \\ l &= 2 \quad (\tfrac{154}{51} \geqslant \eta \geqslant \tfrac{77}{68}), \\ l &= 3 \quad (\tfrac{77}{68} \geqslant \eta \geqslant \tfrac{77}{170}), \end{aligned}$$

and so on, by increasing values of l as the inner radius r_0 of the field shell increases at fixed value of the conductivity ratio β.

The equations of order ϵ^2 correspond closely to those solved in BR. Because the condition (2.17) is satisfied $R^{(1)}$ vanishes. The following expressions are obtained for G_k, where terms of the order γ^2 have been neglected:

$$G_k(z) = [l(l+1) - k(2k+1)]\,\eta^2\gamma(\tfrac{1}{5}z^5 - \tfrac{1}{6}z^3 + \tfrac{1}{6}z)/4! \quad (k \geqslant 1), \tag{7.10a}$$

$$G_0(z) = l(l+1)\,\eta^2\gamma(\tfrac{1}{5}z^5 - \tfrac{1}{6}z^3 + \tfrac{7}{240}z)/4!. \tag{7.10b}$$

Since the mean temperature difference across the spherical fluid layer is fixed, the boundary condition (7.2*b*) assumes the form

$$\overline{\Theta} = 0 \quad \text{at} \quad r = r_0, r_0+1 \tag{7.11}$$

for the spherically averaged part $\overline{\Theta}$ of Θ. For this reason the z-dependence of $G_0(z)$ differs from that of $G_k(z)$ for $k \geqslant 1$. The functions (7.10) can be used to calculate expressions (3.9) and (4.8), with the result

$$M(l,k) = \tfrac{1}{504}\gamma\eta^2 l(l+1)\left(1 - \frac{k(2k+1)}{l(l+1)}\right)^2 \quad (1 \leqslant k \leqslant l), \tag{7.12a}$$

$$M(l,0) = \tfrac{1}{1680}\gamma\eta^2 l(l+1), \tag{7.12b}$$

$$M_2(l) = \frac{1}{l(l+1)}. \tag{7.13}$$

Since $M(l,k)$ is positive for all values of k, $R^{(2)}$ is positive as expected. The individual values for the different solutions discussed in §5 can be easily computed with the help of (6.11) and will not be given here. Instead attention will be focused on the problem of stability.

When (7.12*a*) is used, it is found that $\sigma_4^{(2)}$ and $\sigma_7^{(2)}$ of (6.6) and $\sigma_2^{(2)}$, $\sigma_3^{(2)}$ and $\sigma_5^{(2)}$ of (6.8) are positive, while all other eigenvalues are either negative or vanishing. It must thus be concluded that among the three possible solutions for $l = 3$ only the solution exhibiting the tetrahedral symmetry is physically possible. This is not surprising in view of the fact that its heat transport is a maximum at a given positive value of $R - R_c$ because $R_B^{(2)}$ is smaller than either $R_A^{(2)}$ or $R_C^{(2)}$.

In accordance with the above conclusion it is found that the growing disturbances of solutions A and C tend to transform the respective solutions into the stable solution B. In case A the growing disturbances are

$$\tilde{\alpha}_3 \neq 0, \quad \tilde{\alpha}_i = 0 \quad (i \neq 3), \tag{7.14a}$$

$$\tilde{\beta}_3 \neq 0, \quad \tilde{\beta}_i = 0 \quad (i \neq 3), \tag{7.14b}$$

which indicates that a solution of tetrahedral symmetry of form (5.5d) or the solution turned by 90° about the polar axis is approached. Similarly, in case C the growing disturbances are given by

$$\tilde{\alpha}_0 \neq 0, \quad \tilde{\alpha}_i = 0 \quad (i > 0), \tag{7.15a}$$

$$\tilde{\alpha}_1 \neq 0, \quad \tilde{\alpha}_i = 0 \quad (i \neq 1), \tag{7.15b}$$

$$\tilde{\beta}_1 \neq 0, \quad \tilde{\beta}_i = 0 \quad (i \neq 1), \tag{7.15c}$$

which indicates again that the tetrahedral solution in the description (5.5d) is approached by the disturbance (7.15a) while the disturbances (7.15b, c) indicate a tendency towards the description (5.5c) and the description obtained from (5.5c) by a rotation of 90° about the polar axis.

8. Concluding remarks

One of the unresolved questions posed by the preceding analysis concerns the size of the class of solutions for a given value of l that cannot be obtained by rotations from each other. For $l = 1$ this class contains only one solution. For even values of l each solution of the form (2.10) must be counted twice since a change of the sign of the solution cannot be achieved by a rotation on the sphere. Assuming the general case $R^{(1)} \neq 0$ for even l it was shown in B75 that the cases $l = 2$, 4 and 6 yield 2, 4 and 8 separate solutions respectively. But it has not been rigorously proved that there may not be additional solutions for $l = 4$ and $l = 6$. In this paper it is shown that at least three separate solutions exist for $l = 3$. While the rigorous proof is not available, the existence of an additional solution appears to be rather unlikely. The fact that the extremum principle first derived by Busse (1967) for a plane convection layer is also valid in the spherical case as shown in appendix C implies that there is at least one stable solution since the function (C 1) must have at least one maximum. But in general it cannot be excluded that there is more than one stable solution corresponding to two or more local maxima of the function (C 1). The simultaneous stability of convection rolls and hexagonal convection in a plane layer for a certain range of Rayleigh numbers (Busse 1967) represents such an example. For the case $l = 3$, however, the arguments given at the end of §6 eliminate this possibility.

While the results of this paper except for those of §7 are general in that they require only the spherical symmetry of the basic state, they are limited by the condition that ϵ be small, i.e. only small deviations from the basic state are permitted. But since the analysis has been mainly concerned with symmetry properties, the results are hardly affected by the condition of small ϵ. The symmetry of the solutions cannot be changed by the contributions of higher order in ϵ unless a further bifurcation takes place, i.e. the convection pattern becomes unstable. Because of the exceptional symmetry of the preferred solutions it is expected that those solutions retain their distinction for an extensive range of Rayleigh numbers. Existing numerical solutions do not contradict this conclusion. But the evidence from numerical analysis such as the computation of axisymmetric convection by Schubert & Zebib (1980) is complicated by the fact that patterns of convection appear to be stable which exhibit a symmetry that can only be described by a superposition of spherical harmonics of different degree l. Within the realm of the present analysis it is possible to discuss those cases by assuming that the difference between the values $R^{(0)}$ for l and $l+1$ is of order ϵ^2. But this extension of the problem will not be considered here.

The research reported in this paper has been supported by the Earth Sciences Section of the U.S. National Science Foundation.

Appendix A

By scalar-multiplying the curl of the equation of motion by $\mathbf{r}$ the equation for ψ is obtained:

$$\left(\nabla^2 - P^{-1}\frac{\partial}{\partial t}\right)L_2\psi = -P^{-1}\mathbf{r}\,.\,\nabla\times[\mathbf{u}\times(\nabla\times\mathbf{u})]. \tag{A 1}$$

In this appendix it is shown that the right-hand side of (A 1) vanishes for $\mathbf{u} = \nabla\times(\nabla\times\Phi^{(1)}\mathbf{r})$, where $\Phi^{(1)}$ is of the general form (2.9). Using $\nabla\times\mathbf{u} = \mathbf{r}\times\nabla\nabla^2\Phi^{(1)}$ the following relationshp can be derived:

$$\begin{aligned}\mathbf{r}\,.\,\nabla\times(\mathbf{u}\times(\nabla\times\mathbf{u})) &= \mathbf{r}\,.\,\nabla\times\{[\nabla\times(\nabla\times\mathbf{r}\Phi^{(1)})]\times(\mathbf{r}\times\nabla\nabla^2\Phi^{(1)})\}\\ &= \mathbf{r}\,.\,\nabla\times\{\mathbf{r}\nabla\nabla^2\Phi^{(1)}\,.\,\nabla\times(\nabla\times\mathbf{r}\Phi^{(1)})-\nabla\nabla^2\Phi^{(1)}\,(L_2\Phi^{(1)})\}\\ &= -\mathbf{r}\,.\,\nabla L_2\Phi^{(1)}\times\nabla\nabla^2\Phi^{(1)} = 0.\end{aligned} \tag{A 2}$$

The last equality follows from the fact that the r-dependence of $\Phi^{(1)}$ separates from the (ϕ, θ) – dependence, and the latter is given by a spherical harmonic. Thus the vertical component of vorticity given by $L_2\psi$ is of order ϵ^3 at most.

Appendix B

Those terms in the solvability conditions of (3.1), (3.7), (4.6) and (4.11) that involve ϕ- and θ-derivatives can be transformed into terms without those derivatives with the help of the following relationship:

$$\begin{aligned}2\langle Y_k\nabla_2 w_l\,.\,\nabla_2 w_l^*\rangle &= \langle Y_k(\nabla_2 w_l\,.\,\nabla_2 w_l^* + \nabla_2 w_l^*\,.\,\nabla_2 w_l)\rangle\\ &= 2l(l+1)\langle Y_k w_l w_l^*\rangle - \langle\nabla_2 Y_k\,.\,\nabla_2(w_l w_l^*)\rangle\\ &= [2l(l+1)-k(k+1)]\langle Y_k w_l w_l^*\rangle,\end{aligned} \tag{B 1}$$

where w_l and w_l^* denote two arbitrary spherical harmonics of order l, and Y_k denotes an arbitrary spherical harmonic of degree k. $\nabla_2 w_l$ denotes the two-dimensional gradient of w_l on the unit sphere.

Appendix C

The solvability conditions for the steady solution, (3.2), (3.8) and for the stability equations (4.8), (4.12) can be derived from an extremum principle. In analogy to the corresponding extremum principle for a planar convection layer (Busse 1967) it can be formulated in the following way.

Among all solutions of the form (2.9), (2.10), *those for which the function*

$$F(C_{-l},\ldots,C_l) \equiv (R-R^{(0)})\frac{1}{2}\sum_{m=-l}^{l} C_m^2 + \tfrac{1}{3}M_0(l)\langle w_l w_l w_l\rangle + \frac{1}{4}\sum_{k=0}^{l} M(l,k)\sum_{j=-2k}^{2k}\langle Y_{2k}^j w_l w_l\rangle^2 \quad (\mathrm{C}\ 1)$$

reaches a stationary value correspond to steady solutions of the nonlinear problem in the limit $\epsilon \to 0$. *Those solutions* (2.10) *for which the stationary value is a maximum are stable.* To simplify the notation the following definitions have been used.

$$C_m \equiv \begin{cases} \epsilon\alpha_m & (0 \leqslant m \leqslant l), \\ \epsilon\beta_m & (-l \leqslant m \leqslant -1), \end{cases} \quad (\mathrm{C}\ 2)$$

$$Y_n^{-|m|}(\theta,\phi) \equiv \hat{Y}_n^m(\theta,\phi). \quad (\mathrm{C}\ 3)$$

The necessary condition for a stationary value of function (C 1) is

$$0 = \frac{\partial F}{\partial C_m} = (R-R^{(0)})C_m + M_0(l)\langle w_l w_l Y_l^m\rangle + \sum_{k=0}^{l} M(l,k)\sum_{j=-2k}^{2k}\langle Y_{2k}^j w_l w_l\rangle\langle Y_{2k}^j w_l Y_l^m\rangle \quad (-l \leqslant m \leqslant l), \quad (\mathrm{C}\ 4)$$

which yields the solvability conditions (3.2) and (3.8) at order ϵ^2 and ϵ^3 respectively. It should be noted that the simplifying assumption (2.13) has not been used. The stationary value defined by (C 4) represents a maximum if all eigenvalues of the matrix

$$\frac{\partial^2 F}{\partial C_m\,\partial C_n} = (R-R^{(0)})\delta_{nm} + 2M_0(l)\langle w_l Y_l^n Y_l^m\rangle + \sum_{k=0}^{l} M(l,k)\sum_{j=-2k}^{2k}(2\langle Y_{2k}^j w_l Y_l^n\rangle\langle Y_{2k}^j w_l Y_l^m\rangle + \langle Y_{2k}^j w_l w_l\rangle\langle Y_{2h}^j Y_l^n Y_l^m\rangle) \quad (-l \leqslant n,m \leqslant l) \quad (\mathrm{C}\ 5)$$

are negative–definite. A comparison with (4.7) and (4.12) indicates that the expressions for $M_2\sigma^{(1)}$ and $M_2\sigma^{(2)}$ correspond to the eigenvalues of the matrix (C 5) at order ϵ and ϵ^2 respectively.

REFERENCES

BUSSE, F. H. 1967 The stability of finite amplitude cellular convection and its relation to an extremum principle. *J. Fluid Mech.* **30**, 625–649.

BUSSE, F. H. 1975 Patterns of convection in spherical shells. *J. Fluid Mech.* **72**, 67–85.

BUSSE, F. H. & RIAHI, N. 1980 Nonlinear convection in a layer with nearly insulating boundaries. *J. Fluid Mech.* **96**, 243–256.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Clarendon.

CHOSSAT, P. 1979 Bifurcation and stability of convection flows in a rotating or non rotating spherical shell. *SIAM J. Appl. Math.* **37**, 624–647.

GEIGER, G. 1977 Ph.D. dissertation, University of Munich.

GOLUBITSKY, M. & SCHAEFFER, D. G. 1982 Bifurcations with $O(3)$ symmetry including applications to the Bénard problem. *Commun. Pure Appl. Math.* (in the press).

JOSEPH, D. D. & CARMI, S. 1966 Subcritical convective instability Part 2. Spherical shells. *J. Fluid Mech.* **26**, 796–777.

RIAHI, N., GEIGER, G. & BUSSE, F. H. 1982 Finite Prandtl number convection in spherical shells. *Geophys. Astrophys. Fluid Dyn.* (to be published).

SATTINGER, D. H. 1980 Selection mechanisms in symmetry breaking phenomena. In *Approximation Methods for Navier–Stokes Equations* (ed. R. Rautmann). Lecture Notes in Mathematics, vol. 771, pp. 494–504.

SCHLÜTER, A., LORTZ, D. & BUSSE, F. 1965 On the stability of finite amplitude convection. *J. Fluid Mech.* **23**, 129–144.

SCHUBERT, G. & ZEBIB, A. 1980 Thermal convection of an internally heated infinite Prandtl number fluid in a spherical shell. *Geophys. Astrophys. Fluid Dyn.* **15**, 65–90.

YOUNG, R. E. 1974 Finite amplitude thermal convection in a spherical shell. *J. Fluid Mech.* **63**, 695–721.

ZEBIB, A., SCHUBERT, G. & STRAUS, J. 1980 Infinite Prandtl number thermal convection in a spherical shell. *J. Fluid Mech.* **97**, 257–277.